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The non-dynamical r-matrices of the degenerate Calogero–Moser models

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Abstract. A complete description of the non-dynamical r-matrices of the degenerate Calogero–Moser models based on gl_n is presented. First, the most general momentum-independent r-matrices are given for the standard Lax representation of these systems and those r-matrices whose coordinate dependence can be gauged away are selected. Then the constant r-matrices resulting from the gauge transformation are determined and are related to well known r-matrices. In the hyperbolic/trigonometric case a non-dynamical r-matrix equivalent to a real/imaginary multiple of the Cremmer–Gervais classical r-matrix is found. In the rational case the constant r-matrix corresponds to the antisymmetric solution of the classical Yang–Baxter equation associated with the Frobenius subalgebra of gl_n consisting of the matrices with vanishing last row. These claims are consistent with previous results of Hasegawa and others, which imply that Belavin's elliptic r-matrix and its degenerations appear in the Calogero–Moser models. The advantages of our analysis are that it is elementary and also clarifies the extent to which the constant r-matrix is unique in the degenerate cases.

1. Introduction

The purpose of this paper is to provide a complete description of the non-dynamical, constant r-matrices of the standard Calogero–Moser models [1, 2] associated with degenerate potential functions, which can be obtained by gauge transformations of their usual Lax representation. A preliminary account of part of this work is contained in [3].

The Calogero–Moser-type many-particle systems (for a review, see [4]) have been much studied recently due to their fascinating mathematics and applications ranging from solid state physics to Seiberg–Witten theory. The definition of these models involves a root system and a potential function depending on the inter-particle 'distance'. The potential is given either by the Weierstrass $\mathcal{P}$-function or one of its (hyperbolic, trigonometric or rational) degenerations. The classical equations of motion of the models admit Lax representations,

$$\dot{L} = [L, M] \tag{1}$$

which underlie their integrability. A Lax representation of the Calogero–Moser models based on the root systems of the classical Lie algebras was found by Olshanetsky and Perelomov [5] using symmetric spaces. Recently, new Lax representations for these systems as well as their exceptional Lie algebraic analogues and twisted versions have been constructed [6, 7].

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In general [8], Liouville ingtegrability can be understood as a consequence of the Poisson brackets of the Lax matrix having the r-matrix form,

$$\{L_1, L_2\} = \{L^\mu, L^\nu\}T_\mu \otimes T_\nu = [r_{12}, L_1] - [r_{21}, L_2] \tag{2}$$

where $r_{12} = r^{\mu\nu}T_\mu \otimes T_\nu$ ($r_{21} = r^{\mu\nu}T_\nu \otimes T_\mu$) with some constant matrices T_μ. The components L^μ of the Lax matrix $L = L^\mu T_\mu$ encode the phase space variables, and the components $r^{\mu\nu}$ of the classical r-matrix may, in general, depend on the same variables as $L_1 = L \otimes 1$ and $L_2 = 1 \otimes L$. Of course, L and r also depend on a spectral parameter, in general, but this does not occur for the systems of our interest, and thus is suppressed in (2). When the r-matrix really does depend on the phase space variables, one says that it is 'dynamical'.

The classical r-matrix has been calculated first for the standard Lax representation of the gl_n Calogero–Moser systems associated with degenerate potentials [9], and then for Krichever's [10] spectral-parameter-dependent Lax matrix in the elliptic case [11, 12]. The r-matrices found in these papers are dynamical, but depend only on the coordinates of the particles. These r-matrices have been rederived by means of Hamiltonian reduction in [13, 14], and in a very recent paper [15] they have been generalized explicitly for the BC_n system as well as for all classical Lie algebras. In the physically most interesting gl_n case, dynamical r-matrices have also been found [16–18] for the relativistic deformations of the Calogero–Moser models introduced by Ruijsenaars and Schneider [19]. Then the quantization of the non-relativistic [20] and the relativistic models [21–23] has been investigated in a new framework based on quantum dynamical R-matrices.

The above developments have close connections with the new theory of dynamical r-matrices and associated quantized structures reviewed in [24]. However, since the present understanding of most integrable systems involves constant (i.e. 'non-dynamical') r-matrices, which form a direct link to Poisson–Lie groups and quantum groups [25], it is natural to ask whether the Lax representation of the Calogero–Moser models can be chosen in such a way as to exhibit non-dynamical r-matrices. The obvious way to search for new Lax representations with this property is to perform gauge transformations on the usual Lax representations. In the elliptic case of the standard gl_n models a new Lax representation associated with Belavin's [26] constant elliptic r-matrix has recently been found in this way [27]. To be more precise, the results of [27] are already contained in a somewhat less explicit form in the seminal paper by Hasegawa [28], where the commuting Ruijsenaars operators [29] have been interpreted as commuting transfer matrices based on a realization of the $RLL = LLR$ relation with Belavin's elliptic R-matrix and certain difference L-operators. In fact, the dynamical twisting and the classical and non-relativistic limits of the L-operator leading to Krichever's Lax matrix for the elliptic Calogero–Moser model are indicated in [28] (see also [23]). Then in the paper [30] some delicate limit procedures have been considered, whereby non-dynamical R-matrices can be obtained for the trigonometric degenerations of the Ruijsenaars–Schneider and Calogero–Moser models. The resulting R-matrix was found to be non-unique, one possibility [30] being the spectral-parameter-independent Cremmer–Gervais R-matrix discovered in a different context in [31].

It is clear from the above that Lax representations for the degenerate Calogero–Moser models with non-dynamical r-matrices can be obtained by taking limits of Hasegawa's $RLL = LLR$ relation. However, the details of the admissible limiting procedures appear to be rather complicated and the starting point requires familiarity with quite advanced results. In this circumstance, it might be worthwhile to also understand the possible non-dynamical r-matrices from an elementary viewpoint. This is the objective of the present paper, where we aim to perform a self-contained, systematic analysis of the gauge transformations of the usual Lax representation of the degenerate Calogero–Moser models that lead to constant r-matrices.

The organization and the main results of our work are as follows. First, we describe the most general momentum-independent dynamical r-matrices for the standard Lax representation in section 2. This amounts to a slight but necessary generalization of the Avan–Talon [9] r-matrix as given by theorem 1. Second, we select those dynamical r-matrices that become constant by a gauge transformation (defined by equation (18)) and determine the corresponding 'gauge potentials' $A_k(q)$. This is the content of section 3, in particular proposition 2 and theorem 3. Third, in section 4 we compute explicitly the gauge transformations $g(q)$ (from equation (19)) and the resulting most general constant r-matrix, which is given by theorem 6. It turns out that in the rational case the constant r-matrix is conjugate to the antisymmetric solution of the classical Yang–Baxter equation that belongs to the Frobenius subalgebra of gl_n consisting of the matrices with a vanishing last row [32]. In the hyperbolic/trigonometric cases the sl_n-part of the most general $gl_n \wedge gl_n$-valued constant r-matrix (see proposition 7) is equivalent to a multiple of the Cremmer–Gervais classical r-matrix [31, 33], and it can also be made equal to it by a choice of the gauge transformation. This identification of the constant Calogero–Moser r-matrices is presented in section 5. The main results are summarized once more in the conclusion, which occupies section 6. Except for the notation introduced in section 2, this final section is self-contained and it may be useful to consult it before reading the main text. The details of some proofs are contained in three appendices.

The outcome of our direct analysis of the degenerate Calogero–Moser models is consistent with the previous results [27, 28, 30]. In addition to the advantage that our analysis is elementary, we also clarify the extent to which the constant r-matrix is unique in the degenerate cases. In principle, this uniqueness question cannot be answered by studying the limits of the elliptic case, even though in the final analysis it follows that all our constant r-matrices can be regarded as various degenerations (see also [34]) of Belavin's elliptic r-matrix.

2. Momentum-independent dynamical r-matrices

The standard (degenerate) Calogero–Moser–Sutherland models are defined by the Hamiltonian

$$h = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k<l} v(q_k - q_l) \tag{3}$$

where v is given as

$$v(x) = \begin{cases} x^{-2} & \text{rational case} \\ a^2 \sinh^{-2}(ax) & \text{hyperbolic case} \\ a^2 \sin^{-2}(ax) & \text{trigonometric case.} \end{cases} \tag{4}$$

One has the canonical Poisson brackets $\{p_k, q_l\} = \delta_{k,l}$, the coordinates are restricted to a domain in $\mathbb{R}^n$ where $v(q_k - q_l) < \infty$, and $a > 0$ is a parameter.

Let us fix the following notation for elements of the Lie algebra gl_n:

$$\begin{matrix} H_k := e_{kk} & E_\alpha := e_{kl} \\ H_\alpha := (e_{kk} - e_{ll}) & K_\alpha := (e_{kk} + e_{ll}) \end{matrix} \qquad \text{for} \quad \alpha = \lambda_k - \lambda_l \in \Phi. \tag{5}$$

Here $\Phi = \{(\lambda_k - \lambda_l) | k \neq l\}$ is the set of roots of gl_n, λ_k operates on a diagonal matrix, $H = \mathrm{diag}(H_{1,1}, \ldots, H_{n,n})$ as $\lambda_k(H) = H_{k,k}$ and e_{kl} is the $n \times n$ elementary matrix whose kl-entry is 1. Moreover, we denote the standard Cartan subalgebra of $sl_n \subset gl_n$ as $\mathcal{H}_n$, and put $p = \sum_{k=1}^n p_k H_k$, $q = \sum_{k=1}^n q_k H_k$, $\mathbf{1}_n = \sum_{k=1}^n H_k$.

From the list of known Lax representations we consider the original one [1, 2] for which L is the gl_n-valued function

$$L(q,p) = p + \sqrt{-1}\sum_{\alpha\in\Phi} w(\alpha(q))E_\alpha \tag{6}$$

where the real function w is chosen according to

$$w(x) = \begin{cases} x^{-1} \\ a\sinh^{-1}(ax) \\ a\sin^{-1}(ax). \end{cases} \tag{7}$$

Then the function

$$F := -\frac{w'}{w} \tag{8}$$

enjoys the important identities

$$F' = -w^2 \tag{9}$$

$$F(x) + F(y) = \frac{w(x)w(y)}{w(x+y)} \tag{10}$$

$$F(x-y)\,(F(x) - F(y)) + F(x)F(y) = \mathcal{B} \tag{11}$$

where, respectively to the cases above,

$$\mathcal{B} = \begin{cases} 0 \\ a^2 \\ -a^2. \end{cases} \tag{12}$$

For any real function f (such as v, w or F), we introduce the functions f_k and f_α of q as

$$f_k(q) := f(q_k) \qquad f_\alpha(q) = f(\alpha(q)) \tag{13}$$

and sometimes write f_{kl} for f_α if $\alpha = (\lambda_k - \lambda_l)$. As an $n\times n$ matrix $L_{k,l} = p_k\delta_{k,l} + \sqrt{-1}(1 - \delta_{k,l})w(q_k - q_l)$, but L can also be used in any other representation of gl_n. The r-matrix corresponding to this L was studied by Avan and Talon [9], who found that it is necessarily dynamical, and may be chosen so as to depend on the coordinates q_k only. We next describe a slight generalization of their result.

Theorem 1. *The most general $gl_n \otimes gl_n$-valued r-matrix that satisfies (2) with the Lax matrix in (6) and depends only on q is given by*

$$r(q) = -\sum_{\alpha\in\Phi} F_\alpha(q)E_\alpha\otimes E_{-\alpha} + \tfrac{1}{2}\sum_{\alpha\in\Phi} w_\alpha(q)(C_\alpha(q) - K_\alpha)\otimes E_\alpha + \mathbf{1}_n\otimes Q(q) \tag{14}$$

where the $C_\alpha(q)$ are $\mathcal{H}_n \subset sl_n$-valued functions subject to the conditions

$$C_{-\alpha}(q) = -C_\alpha(q) \qquad \beta(C_\alpha(q)) = \alpha(C_\beta(q)) \qquad \forall\alpha,\beta\in\Phi \tag{15}$$

and $Q(q)$ is an arbitrary gl_n-valued function.

Remarks. The functions C_α can be given arbitrarily for the simple roots, and are then uniquely determined for the other roots by (15). The r-matrix found by Avan and Talon [9] is recovered from (14) with $C_\alpha \equiv 0$; and we refer to $r(q)$ in (14) as the *Avan–Talon r-matrix in its general form.* Given that this holds for the Avan–Talon r-matrix, the fact that $r(q)$ above satisfies (2) with any $Q(q)$ and $C_\alpha(q)$ subject to (15) is easy to verify. Theorem 1 can be proved by a careful calculation along the lines of [12]. For the details, see appendix A.

3. Is $r(q)$ gauge equivalent to a constant?

A gauge transformation of a given Lax representation (1) has the form

$$L \mapsto L' = gLg^{-1} \qquad M \mapsto M' = gMg^{-1} - \frac{\mathrm{d}g}{\mathrm{d}t}g^{-1} \tag{16}$$

where g is an invertible matrix function on the phase space. If L satisfies (2), then L' will have similar Poisson brackets with a transformed r-matrix r'. The question now is whether one can remove the q dependence of any of the r-matrices in (14) by a gauge transformation. It is natural to assume this gauge transformation to be p-independent, i.e. defined by some function $g : q \mapsto g(q) \in GL_n$. In this case we find that

$$\{L'_1, L'_2\} = [r'_{12}, L'_1] - [r'_{21}, L'_2] \tag{17}$$

holds with

$$r'(q) = (g(q) \otimes g(q)) \left(r(q) + \sum_{k=1}^{n} A_k(q) \otimes H_k \right) (g(q) \otimes g(q))^{-1} \tag{18}$$

$$A_k(q) := -g^{-1}(q)\partial_k g(q) \qquad \partial_k := \frac{\partial}{\partial q_k}. \tag{19}$$

The meaning of this formula is that if $r(q)$ is the most general p-independent r-matrix for which L (6) satisfies (2), then $r'(q)$ has the analogous property in relation to L'.

We wish to find $r(q)$ and $g(q)$ such that $\partial_k r' = 0$. On account of (18) this is equivalent to

$$\partial_k \left(r + \sum_{l=1}^{n} A_l \otimes H_l \right) + \left[r + \sum_{l=1}^{n} A_l \otimes H_l, A_k \otimes \mathbf{1}_n + \mathbf{1}_n \otimes A_k \right] = 0. \tag{20}$$

By using (19), whereby

$$\partial_k A_l - \partial_l A_k + [A_l, A_k] = 0 \tag{21}$$

it is useful to rewrite (20) as

$$\partial_k r + \sum_{l=1}^{n} \partial_l A_k \otimes H_l + [r, A_k \otimes \mathbf{1}_n + \mathbf{1}_n \otimes A_k] + \sum_{l=1}^{n} A_l \otimes [H_l, A_k] = 0. \tag{22}$$

Our strategy is to first find $A_k(q)$ and $r(q)$ from equations (21), (22), and then determine $g(q)$ and the resulting constant r-matrix. For this we now parametrize A_k as

$$A_k(q) = \sum_{l=1}^{n} A_k^l(q) H_l + \sum_{\alpha\in\Phi} A_k^\alpha(q) E_\alpha \tag{23}$$

and expand the r-matrix from theorem 1 in the form

$$r(q) = -\sum_{\alpha} F_\alpha(q) E_\alpha \otimes E_{-\alpha} + \sum_{i,\alpha} r_i^\alpha(q) H_i \otimes E_\alpha + \sum_{i} Q^i(q) \mathbf{1}_n \otimes H_i. \tag{24}$$

Here we have

$$r_i^\alpha(q) = Q^\alpha(q) + \tfrac{1}{2} w_\alpha(q)\, \mathrm{tr}\,(H_i(C_\alpha(q) - K_\alpha)) \tag{25}$$

$$Q(q) = \sum_{i=1}^{n} Q^i(q) H_i + \sum_{\alpha\in\Phi} Q^\alpha(q) E_\alpha \tag{26}$$

where $Q(q)$, $C_\alpha(q)$ and K_α appear in (14).

With reference to the conventions (5), we define the structure constants $c^{\alpha+\beta}_{\alpha,\beta}$ by writing $[E_\alpha, E_\beta] = c^{\alpha+\beta}_{\alpha,\beta} E_{\alpha+\beta}$ if α, β, $(\alpha+\beta)$ all belong to Φ, and $c^{\alpha+\beta}_{\alpha,\beta} := 0$ otherwise. Then (21) yields

$$\partial_l A^i_k - \partial_k A^i_l = \sum_{\alpha\in\Phi} \alpha_i A^\alpha_l A^{-\alpha}_k \qquad \forall i, k, l \tag{27}$$

$$\partial_l A^\alpha_k - \partial_k A^\alpha_l = \sum_{i=1}^n \alpha_i (A^i_l A^\alpha_k - A^i_k A^\alpha_l) + \sum_{\gamma\in\Phi} c^\alpha_{\gamma,\alpha-\gamma} A^\gamma_l A^{\alpha-\gamma}_k \qquad \forall\alpha \quad \forall k, l. \tag{28}$$

The $H_i \otimes H_j$ and $H_i \otimes E_\alpha$ components of (22) require that

$$\partial_k Q^j + \partial_j A^i_k + \sum_{\alpha\in\Phi} \alpha_j r^\alpha_i A^{-\alpha}_k = 0 \qquad \forall i, j, k \tag{29}$$

$$\partial_k r^\alpha_i - \alpha_i F_\alpha A^\alpha_k + \sum_{j=1}^n \alpha_j Q^j A^\alpha_k - \sum_{j=1}^n \alpha_j A^j_k r^\alpha_i + \sum_{\gamma\in\Phi} c^\alpha_{\gamma,\alpha-\gamma} r^\gamma_i A^{\alpha-\gamma}_k + \sum_{j=1}^n \alpha_j A^i_j A^\alpha_k = 0 \qquad \forall i, k, \alpha. \tag{30}$$

From the $E_\alpha \otimes H_i$ and $E_\alpha \otimes E_\beta$ components of (22) we find that

$$\partial_i A^\alpha_k + \alpha_i F_\alpha A^\alpha_k = 0 \qquad \forall i, k, \alpha \tag{31}$$

$$\delta_{\beta,-\alpha}\alpha_k w^2_\alpha - c^{\alpha+\beta}_{\alpha,\beta} \frac{w_\alpha w_\beta}{w_{\alpha+\beta}} A^{\alpha+\beta}_k + \sum_{j=1}^n \alpha_j r^\beta_j A^\alpha_k + \sum_{j=1}^n \beta_j A^\alpha_j A^\beta_k = 0 \qquad \forall k, \alpha, \beta. \tag{32}$$

Note that to derive (32) we have used the identities (9) and (10), and the symmetry properties of the structure constants.

It is convenient to focus first on the last two equations, since they do not contain the Cartan components of A_k. Equation (31) obviously implies that

$$A^\alpha_k(q) = w_\alpha(q) b^\alpha_k \qquad b^\alpha_k : \text{some constants.} \tag{33}$$

The constants are then determined as follows.

Proposition 2. *Equation (32) admits solution for the constants b^α_k only for those two families of $r(q)$ in (14) for which the C_α are chosen according to*

$$\textit{case I:} \quad C_\alpha = -H_\alpha \quad \forall\alpha\in\Phi \qquad \textit{or} \qquad \textit{case II:} \quad C_\alpha = H_\alpha \quad \forall\alpha\in\Phi. \tag{34}$$

For $\alpha = \lambda_m - \lambda_l$, the b^α_k are given, respectively, by

$$b^{(\lambda_m-\lambda_l)}_k = \delta_{km} + \Omega \quad \textit{in case I} \qquad \textit{and} \qquad b^{(\lambda_m-\lambda_l)}_k = \delta_{kl} + \Omega \quad \textit{in case II} \tag{35}$$

where Ω is an arbitrary constant.

Proof. The statement is obtained by an elementary, but rather lengthy inspection of equation (32). This is contained in appendix B. □

It is easy to explain why we obtained *two* series of solutions in the above. Namely, they arise due to the fact that L in (6) is a self-adjoint matrix. Indeed, $L^\dagger = L$ implies that if $r(q)$ solves (2) then $r^\dagger(q)$ also solves it, where $(u_1 \otimes u_2)^\dagger = u_1^\dagger \otimes u_2^\dagger$. Furthermore, if $r(q)$ is gauge transformed to a constant r' by $g(q)$, then $r^\dagger(q)$ is transformed to $(r')^\dagger$ by $(g^\dagger)^{-1}$. The two series of solutions described in proposition 2 are exchanged by this symmetry. It is thus enough to consider only one of these series, and from now on we concentrate on case I.

As the main result of this section, we now give the most general 'gauge potential' A_k and $r(q)$ for which r' (18) will be constant.

Theorem 3. *The most general solution of equations (21) and (22) for A_k and Q in case I of proposition 2 can be described as follows. The root part of A_k is determined by proposition 2, while its Cartan part has the form*†

$$A_k^l = F_{\lambda_l-\lambda_k} + \Omega \sum_{m\ (m\neq l)} F_{\lambda_l-\lambda_m} + \partial_k\theta \qquad (\forall k, l = 1, \dots, n) \tag{36}$$

where $\theta(q)$ is an arbitrary smooth function. The function $Q(q) \in gl_n$ is given by

$$Q = -\sum_{k=1}^{n} A_k^k H_k - \Omega \sum_{\alpha\in\Phi} w_\alpha E_\alpha + g^{-1} Q' g \tag{37}$$

where $g(q) \in GL_n$ denotes a solution of $\partial_k g = -g A_k$ and $Q' \in gl_n$ is an arbitrary constant.

Proof. The main steps of the proof can be outlined as follows. After choosing case I of proposition 2, the right-hand side of (27) can be calculated. The general solution of (27) for the unknowns A_k^l is then found to be

$$A_k^l = F_{\lambda_l-\lambda_k} + \Omega \sum_{m\ (m\neq l)} F_{\lambda_l-\lambda_m} + \partial_k\theta^l \qquad (\forall k, l = 1, \dots, n) \tag{38}$$

where the θ^l are arbitrary smooth functions of q. Next, it is verified that (38) solves (28) if and only if

$$\theta^1 = \theta^2 = \cdots = \theta^n := \theta. \tag{39}$$

At this point we have the general solution for A_k and the remaining task is to solve (29) and (30) for Q. By also using (25) with $C_\alpha = -H_\alpha$, these are inhomogeneous linear differential equations for Q. It is an easy matter to check that (37) with $Q' = 0$ gives a particular solution, and that the difference δQ of two solutions must satisfy the equations

$$\partial_k(\delta Q) + [\delta Q, A_k] = 0 \qquad (\forall k = 1, \dots, n). \tag{40}$$

The proof is completed by noting that the last equation is equivalent to $\partial_k(g(\delta Q)g^{-1}) = 0$ with $\partial_k g = -g A_k$. □

We wish to make some observations on the above result. Firstly, note that if r' is the constant r-matrix obtained from (18) in the case

$$\theta = 0 \qquad Q' = 0 \tag{41}$$

then in the general case of theorem 3 the same formula yields

$$r' + \mathbf{1}_n \otimes Q'. \tag{42}$$

This means that the free parameters θ and Q' in (36) and (37) are irrelevant. Henceforth they will be set to zero. An additional convenience of this choice is that it guarantees the antisymmetry of r' (18). In fact, one can compute the symmetric part of $(r + \sum_k A_k \otimes H_k)$ and finds it to be zero if $Q' = 0$. Secondly, it is worth pointing out that

$$r' \in sl_n \wedge sl_n \quad \Leftrightarrow \quad \Omega = -\frac{1}{n}. \tag{43}$$

† Note that $F_{\lambda_l-\lambda_l} = 0$ by the definition of $F_{\lambda_l-\lambda_k}$ in (13).

Indeed, the condition $r' \in sl_n \wedge sl_n$ is clearly equivalent to $(r + \sum_k A_k \otimes H_k) \in sl_n \wedge sl_n$, and this is easily calculated to hold if and only if $Q' = 0$ and $\Omega = -\frac{1}{n}$. Since for a given A_k the solution of (19) for $g(q) \in GL_n$ is unique up to a constant,

$$g(q) \to g_0 g(q) \qquad \forall g_0 \in GL_n \tag{44}$$

we can also conclude that if the condition $r' \in sl_n \otimes sl_n$ is imposed, then r' is necessarily antisymmetric and is uniquely determined up to an automorphism of sl_n.

Finally, let us observe that our $r(q)$ and $A_k(q)$ for which r' will be a constant admit the interesting decompositions

$$r = \tilde{r} - \Omega \mathbf{1}_n \otimes \mathcal{A} \qquad A_k = \tilde{A}_k + \Omega \mathcal{A} \tag{45}$$

where

$$\mathcal{A} = \sum_{l,m\,(l \neq m)} \left(F_{\lambda_l - \lambda_m} H_l + w_{\lambda_l - \lambda_m} E_{\lambda_l - \lambda_m} \right). \tag{46}$$

Here $r(q)$, A_k are given by theorem 3 together with (41). In the rest of the paper we shall determine the corresponding constant r-matrices from (18). It will be convenient to consider first the $\Omega = 0$ special case, for which r, A_k, r' become $\tilde{r}$, $\tilde{A}_k$, $\tilde{r}'$, respectively.

4. Constant r-matrices from gauge transformation

If A_k is given so that (21) holds then the gauge transformation $g(q)$ can be determined from the differential equation in (19). By taking A_k and $r(q)$ from theorem 3 with (41), this g will transform the dynamical r-matrix $r(q)$ into an antisymmetric constant (18). Here we shall determine $g(q)$ and r' explicitly. For an antisymmetric constant r' the (modified) classical Yang–Baxter equation is a sufficient condition for the Jacobi identity $\{\{L'_1, L'_2\}, L'_3\}$+cycl. $= 0$, which will be seen to hold for the r-matrices found below.

4.1. The case of $\Omega = 0$

Now we calculate the gauge transformation and the resulting constant r-matrix in the special case of theorem 3 for which $\Omega = 0$ and (41) hold. In agreement with (45), the various quantities will carry a tilde in this case. We shall use the notation

$$I_k^n := \{1, \ldots, n\} \setminus \{k\} \qquad \forall k = 1, \ldots, n \tag{47}$$

and write the elements of gl_n as matrices. Then $\tilde{r}(q)$ and $\tilde{A}_k(q)$ take the following form:

$$\tilde{r} = -\sum_{1 \leqslant k \neq l \leqslant n} (F_{kl} e_{kl} \otimes e_{lk} + w_{kl} e_{kk} \otimes e_{kl}) \qquad \tilde{A}_k = \sum_{l \in I_k^n} (w_{kl} e_{kl} + F_{lk} e_{ll}). \tag{48}$$

Let us start by defining the matrix function φ of q as follows: $\varphi_{nk} := 1$ for any $k = 1, \ldots, n$ and

$$\varphi_{jk} := \sum_{\substack{P \subset I_k^n \\ |P| = n-j}} \left(\prod_{l \in P} F_l \right) \qquad \forall k \quad 1 \leqslant j \leqslant n-1 \tag{49}$$

where $|P|$ denotes the number of elements of P. Moreover, let χ be the $n \times n$ matrix function of q given by

$$\chi_{jk} = \delta_{jk} \prod_{l \in I_k^n} \frac{1}{w_l}. \tag{50}$$

These formulae yield invertible matrices on the admissible domain of q, where $v(q)$ is finite. This is obvious for the diagonal matrix χ. By using the identity

$$\sum_{l=1}^{n}(-F_i)^{l-1}\varphi_{lj} = \prod_{\tau\in I_j^n}(F_\tau - F_i) \tag{51}$$

we can also find the inverse of φ explicitly

$$\left(\varphi^{-1}\right)_{jk} = (-F_j)^{k-1}\prod_{l\in I_j^n}\frac{1}{(F_l - F_j)}. \tag{52}$$

Proposition 4. *A gauge transformation $\tilde{g}(q) \in GL_n$ that satisfies*

$$\partial_k\tilde{g}(q) = -\tilde{g}(q)\tilde{A}_k(q) \tag{53}$$

with $\tilde{A}_k$ in (48) is given by $\tilde{g}(q) = \varphi(q)\chi(q)$, where φ and χ are defined by (49) and (50).

Proof. The componentwise form of (53) with $\tilde{A}_k$ in (48) reads

$$\partial_k\tilde{g}_{ik} = 0 \qquad \forall i,k\in\{1,\ldots,n\} \tag{54}$$

$$\partial_k\tilde{g}_{ij} = -\tilde{g}_{ij}F_{jk} - \tilde{g}_{ik}w_{kj} \qquad \forall i,j,k\in\{1,\ldots,n\} \quad j\neq k. \tag{55}$$

We note that the matrix

$$\tilde{g}_{ij}(q) = \prod_{l\in I_j^n}\frac{1}{w(q_l+c_i)} \qquad i,j\in\{1,\ldots,n\} \tag{56}$$

where the $\{c_i\}_{i=1}^n$ are pairwise distinct constants, yields a solution to these equations. Indeed, (54) holds obviously, while (55) is checked with the aid of the identity (10). Using (10) again, we can rewrite the matrix $\tilde{g}(q)$ defined by (56) in the product form

$$\tilde{g}(q) = \mathbb{C}\varphi(q)\chi(q) \tag{57}$$

where $\mathbb{C}$ is the invertible constant matrix given by

$$\mathbb{C}_{ij} = \frac{1}{w(c_i)^{n-1}}\left(F(c_i)\right)^{j-1}. \tag{58}$$

Since equation (53) determines $\tilde{g}$ up to multiplication by a constant matrix from the left, the required statement follows. □

We can now calculate the gauge-transformed r-matrix from (18). The result turns out to be an antisymmetric, constant solution of the (modified) classical Yang–Baxter equation,

$$[r'_{12}, r'_{13}] + [r'_{12}, r'_{23}] + [r'_{13}, r'_{23}] = -\mathcal{B}\hat{\mathcal{F}} \tag{59}$$

where $\mathcal{B}$ appears in (12) and $\hat{\mathcal{F}} \in (gl_n)^{3\wedge}$ is given by

$$\hat{\mathcal{F}} := \sum_{i,j,k,l,r,s=1}^{n}\mathcal{F}_{ij,kl}^{rs}e_{ji}\otimes e_{lk}\otimes e_{rs} \qquad \text{with} \quad [e_{ij},e_{kl}] = \sum_{r,s=1}^{n}\mathcal{F}_{ij,kl}^{rs}e_{rs}. \tag{60}$$

Proposition 5. *The gauge transform of $\tilde{r}(q)$ in (48) by $\tilde{g}(q)$ in proposition 4 is given by*

$$\tilde{r}' = \sum_{(a,b,c,d)\in S}\left(\mathcal{B}e_{ab}\wedge e_{cd} - e_{a+1,b}\wedge e_{c+1,d}\right) \tag{61}$$

$$S = \{\,(a,b,c,d)\in\mathbb{N}^4 \mid a+c+1 = b+d,\ 1\leqslant b\leqslant a<n,\ b\leqslant c<n,\ 1\leqslant d\leqslant n\}.$$

This formula defines an antisymmetric solution of (59).

Proof. The first statement is verified by a direct calculation, which is described in appendix C. The fact that $\tilde{r}'$ solves (59) can also be checked directly. Alternatively, it follows from the identification of $\tilde{r}'$ in terms of certain well known solutions of (59), which is presented in section 5. □

It is clear from (59) that the two terms in (61) must separately satisfy the classical Yang–Baxter equation,

$$[b_{12}, b_{13}] + [b_{12}, b_{23}] + [b_{13}, b_{23}] = 0. \tag{62}$$

In fact, this holds since the first term

$$b_{gl_n} := \sum_{(a,b,c,d)\in S} e_{ab} \wedge e_{cd} \tag{63}$$

is nothing but the classical r-matrix associated with the Frobenius subalgebra of gl_n spanned by the matrices with vanishing last row, which is described as an example in [32]. More explicitly, it reads as

$$b_{gl_n} = \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} e_{jj} \wedge e_{n-k,n+1-k} + \sum_{1\leqslant i<j\leqslant n}\sum_{m=1}^{j-i-1} e_{n+1-i-m,n+1-j} \wedge e_{n+m-j,n+1-i}. \tag{64}$$

The second term is a transform of the first one according to

$$\sum_{(a,b,c,d)\in S} e_{a+1,b} \wedge e_{c+1,d} = -(\sigma \otimes \sigma) b_{gl_n} \tag{65}$$

where $\sigma : gl_n \to gl_n$ is the inner automorphism

$$\sigma : e_{ij} \mapsto e_{n+1-i,n+1-j}. \tag{66}$$

Finally, we note for later purposes that

$$\tilde{r}' = \mathcal{B} b_{gl_n} + (\sigma \otimes \sigma) b_{gl_n} = \tilde{r}'_{sl_n} + X \wedge \mathbf{1}_n \tag{67}$$

where $\tilde{r}'_{sl_n} \in sl_n \wedge sl_n$ and

$$X = -\frac{1}{n}\sum_{k=1}^{n-1}(n-k) e_{k+1,k} - \frac{\mathcal{B}}{n}\sum_{k=1}^{n-1} k e_{k,k+1}. \tag{68}$$

Of course, $\tilde{r}'_{sl_n}$ satisfies the same equation (59) as $\tilde{r}'$.

4.2. The case of an arbitrary Ω

Now we tackle the general case by making use of the decompositions of $r(q)$ and A_k in (45).

It is natural to look for $g(q)$ as a product

$$g(q) = h(q)\tilde{g}(q) \tag{69}$$

where $\tilde{g}(q)$ is given in proposition 4. Then the equation $\partial_k g = -(\tilde{A}_k + \Omega\mathcal{A})g$ reduces to

$$\partial_k h = -h\tilde{\mathcal{A}}\Omega \qquad \text{with} \quad \tilde{\mathcal{A}} := \tilde{g}\mathcal{A}\tilde{g}^{-1} \tag{70}$$

where $\mathcal{A}$ is given in (46). By also using the decomposition of $r(q)$ in (45) we obtain from (18) that

$$r' = (h(q) \otimes h(q))\big(\tilde{r}' + \Omega\tilde{\mathcal{A}}(q) \wedge \mathbf{1}_n\big)(h(q) \otimes h(q))^{-1} \tag{71}$$

where $\tilde{r}'$ is given by (61). The fact that r' and $\tilde{r}'$ are both constant permits us to prove the following result without further explicit calculation.

Theorem 6. *With the above notations and $\tilde{r}'$, X defined in (61) and (67), we have*

$$h(q) = g_0 \exp\left(-Xn\Omega\sum_{i=1}^{n} q_i\right) \tag{72}$$

where $g_0 \in GL_n$ is an arbitrary constant, and

$$r' = (g_0 \otimes g_0)\left(\tilde{r}'_{sl_n} + (n\Omega + 1)X \wedge \mathbf{1}_n\right)(g_0 \otimes g_0)^{-1} \tag{73}$$

is the most general constant r-matrix resulting from gauge transformation.

Proof. By substituting (67), we can rewrite (71) as the sum $r' = r'_{sl_n} + r'_{\text{rest}}$ with

$$r'_{sl_n} = (h(q) \otimes h(q))\tilde{r}'_{sl_n}(h(q) \otimes h(q))^{-1} \tag{74}$$

and

$$r'_{\text{rest}} = \left(h(q)(\Omega\tilde{\mathcal{A}}(q) + X)h^{-1}(q)\right) \wedge \mathbf{1}_n. \tag{75}$$

Since r' is constant, these two terms must be constant separately. Recall now that $\tilde{\mathcal{A}}(q)$ is independent of Ω by its definition (70) and that for $\Omega = -\frac{1}{n}$ we must have $r' \in sl_n \wedge sl_n$ (43). This implies that $(X - \frac{1}{n}\tilde{\mathcal{A}}(q))$ must vanish, whereby

$$\tilde{\mathcal{A}} = nX. \tag{76}$$

Hence we obtain (72) from the differential equation in (70). However, then the fact that r'_{sl_n} is constant shows that the relation

$$[X \otimes \mathbf{1}_n + \mathbf{1}_n \otimes X, \tilde{r}'_{sl_n}] = 0 \tag{77}$$

which is equivalent to

$$r'_{sl_n} = (g_0 \otimes g_0)\tilde{r}'_{sl_n}(g_0 \otimes g_0)^{-1} \tag{78}$$

must be valid. By substituting these results back into (71) we arrive at (73). □

Incidentally, we have also verified by explicit calculation that (76) and (77) are indeed satisfied, which represents a reassuring check on the foregoing considerations in the paper.

5. Identification of the constant r-matrices

The constant r-matrix (73) is a solution of (59). For the rational Calogero–Moser model, $\mathcal{B} = 0$, this is the classical Yang–Baxter equation. In this case the identification of the r-matrix in terms of a Frobenius subalgebra of gl_n has already been mentioned (67). In the hyperbolic/trigonometric cases (59) is the modified classical Yang–Baxter equation, whose antisymmetric solutions have been classified by Belavin and Drinfeld [32] for the complex simple Lie algebras. A well known solution for the Lie algebra sl_n, with the normalization

$$[\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}] = -\hat{\mathcal{F}} \tag{79}$$

is the so-called Cremmer–Gervais classical r-matrix, which we quote from [33] as

$$r_{\text{CG}} = \sum_{1\leqslant i<j\leqslant n} e_{ij} \wedge e_{ji} + 2\sum_{1\leqslant i<j\leqslant n}\sum_{m=1}^{j-i-1} e_{i,j-m} \wedge e_{j,i+m}$$
$$+\frac{1}{n}\sum_{1\leqslant i<j\leqslant n}(n + 2(i-j))e_{ii} \wedge e_{jj}. \tag{80}$$

Note that $r_{\mathrm{CG}} \in sl_n \wedge sl_n$ and $\hat{\mathcal{F}}$ (60) belongs to $(sl_n)^{3\wedge}$. Below we show that for $\mathcal{B} \neq 0$ the sl_n-part of the constant Calogero–Moser r-matrix (73) is equivalent to r_{CG}.

We shall need the following properties of r_{CG}. As in [33], first introduce J_0, $J_\pm \in sl_n$ by

$$J_0 = \frac{1}{2}\sum_{k=1}^{n}(n+1-2k)e_{kk} \qquad J_+ = \sum_{k=1}^{n-1}(n-k)e_{k,k+1} \qquad J_- = \sigma(J_+) = \sum_{k=1}^{n-1} k e_{k+1,k}. \tag{81}$$

They generate the principal sl_2 subalgebra of sl_n,

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0. \tag{82}$$

Then define the elements $b^\pm_{\mathrm{CG}} := \mp\frac{1}{2}[J_\pm \otimes \mathbf{1}_n + \mathbf{1}_n \otimes J_\pm, r_{\mathrm{CG}}] \in sl_n \wedge sl_n$. Explicitly,

$$b^+_{\mathrm{CG}} = \sum_{k=1}^{n-1} d_k \wedge e_{k,k+1} + \sum_{1\leqslant i<j\leqslant n}\sum_{m=1}^{j-i-1} e_{i,j-m+1} \wedge e_{j,i+m} \qquad d_k := \sum_{j=1}^{k} e_{jj} - \frac{k}{n}\mathbf{1}_n. \tag{83}$$

On account of $(\sigma\otimes\sigma)r_{\mathrm{CG}} = -r_{\mathrm{CG}}$, with σ defined in (66), $b^-_{\mathrm{CG}} = (\sigma\otimes\sigma)b^+_{\mathrm{CG}}$. It has been pointed out in [33] that the subspace of $sl_n \wedge sl_n$ spanned by r_{CG} and $b^\pm_{\mathrm{CG}}$ is an irreducible representation of the principal sl_2 subalgebra. In fact, for the operators

$$\mathcal{J}_{0,\pm}(Y) := [J_{0,\pm} \otimes \mathbf{1}_n + \mathbf{1}_n \otimes J_{0,\pm}, Y] \qquad \forall Y \in gl_n \otimes gl_n \tag{84}$$

one has the relations

$$\begin{aligned}
\mathcal{J}_0 \begin{pmatrix} b^+_{\mathrm{CG}} \\ r_{\mathrm{CG}} \\ b^-_{\mathrm{CG}} \end{pmatrix} &= \begin{pmatrix} b^+_{\mathrm{CG}} \\ 0 \\ -b^-_{\mathrm{CG}} \end{pmatrix} \\
\mathcal{J}_+ \begin{pmatrix} b^+_{\mathrm{CG}} \\ r_{\mathrm{CG}} \\ b^-_{\mathrm{CG}} \end{pmatrix} &= \begin{pmatrix} 0 \\ -2b^+_{\mathrm{CG}} \\ r_{\mathrm{CG}} \end{pmatrix} \\
\mathcal{J}_- \begin{pmatrix} b^+_{\mathrm{CG}} \\ r_{\mathrm{CG}} \\ b^-_{\mathrm{CG}} \end{pmatrix} &= \begin{pmatrix} -r_{\mathrm{CG}} \\ 2b^-_{\mathrm{CG}} \\ 0 \end{pmatrix}.
\end{aligned} \tag{85}$$

It follows from these relations that $b^\pm_{\mathrm{CG}}$ satisfy the classical Yang–Baxter equation [33], and the identification of $b^\pm_{\mathrm{CG}}$ in terms of Frobenius subalgebras of sl_n is also described in this reference.

Now we are prepared to establish the connection between r_{CG} and the r-matrix r' (73). The key observation is the following identity:

$$-(T\otimes T)\tilde{r}'_{sl_n} = b^+_{\mathrm{CG}} + \mathcal{B}b^-_{\mathrm{CG}} \tag{86}$$

where $T : gl_n \to gl_n$ denotes matrix transposition. This can be checked directly from the formulae (67), (64) and (83). It permits us to transform $\tilde{r}'_{sl_n}$ into a multiple of r_{CG} in a simple manner. To treat the hyperbolic/trigonometric cases together, we introduce the parameter

$$a' = \begin{cases} a & \text{hyperbolic case} \\ \sqrt{-1}a & \text{trigonometric case} \end{cases} \tag{87}$$

whose square $\mathcal{B} = (a')^2$ appears in (59). By using (85) it is not difficult to verify that

$$(u_- u_+ \otimes u_- u_+) \left(T \otimes T \tilde{r}'_{sl_n}\right) (u_- u_+ \otimes u_- u_+)^{-1} = a' r_{\mathrm{CG}} \tag{88}$$

with

$$u_- := \exp\left(\frac{a'}{2} J_-\right) \qquad u_+ := \exp\left(-\frac{1}{a'} J_+\right). \tag{89}$$

According to (88) the sl_n-part of r' is equivalent to $a' r_{\mathrm{CG}}$ under a Lie algebra automorphism.

In the end, note from (68) and (81) that

$$X = -\frac{1}{n}(J_+^T + \mathcal{B} J_-^T). \tag{90}$$

This allows us to present the r-matrix associated with

$$L'(q,p) = g_0 h(q) \tilde{g}(q) L(q,p) (g_0 h(q) \tilde{g}(q))^{-1} \tag{91}$$

in a 'standard form'. Here $h(q)$ and $\tilde{g}(q)$ are the same as in theorem 6, and our final result is formulated as follows.

Proposition 7. *Consider the hyperbolic/trigonometric Calogero–Moser models. If in theorem 6 the constant g_0 is chosen to be*

$$g_0 = \exp\left(-\frac{a'}{2} J_-^T\right) \exp\left(\frac{1}{a'} J_+^T\right) \tag{92}$$

then the r-matrix (73) becomes

$$r' = a'(T \otimes T)\left(r_{\mathrm{CG}} + 2\left(\Omega + \frac{1}{n}\right) J_0 \wedge \mathbf{1}_n\right). \tag{93}$$

Proof. By means of the sl_2 algebra (82) and (90) it is easy to check that $g_0 X g_0^{-1} = \frac{2a'}{n} J_0^T$. The statement is obtained by combining this with (88). □

This proposition describes the precise relationship between the most general constant r-matrices of the hyperbolic/trigonometric Calogero–Moser models and the standard Cremmer–Gervais classical r-matrices.

6. Conclusion

In this paper we have determined the most general constant r-matrices that may be obtained by coordinate-dependent gauge transformations of the standard Lax representation (6) of the degenerate Calogero–Moser models associated with gl_n. Up to automorphisms of gl_n (i.e. up to conjugation by constants $g_0 \in GL_n$ and transpose) and addition of an irrelevant term $\mathbf{1}_n \otimes Q'$ with any constant $Q' \in gl_n$, the most general such r-matrix turned out to have the form

$$r' = \sum_{(a,b,c,d) \in S} \left(\mathcal{B} e_{ab} \wedge e_{cd} - e_{a+1,b} \wedge e_{c+1,d}\right) + n\Omega X \wedge \mathbf{1}_n \tag{94}$$

where

$$X = -\frac{1}{n}\sum_{k=1}^{n-1}(n-k) e_{k+1,k} - \frac{\mathcal{B}}{n}\sum_{k=1}^{n-1} k e_{k,k+1}$$

$\mathcal{B}$ is given according to (12) in correspondence with the rational, hyperbolic and trigonometric potential functions (4), Ω is an arbitrary constant scalar, and

$$S = \{ (a,b,c,d) \in \mathbb{N}^4 \mid a+c+1 = b+d,\ 1 \leqslant b \leqslant a < n,\ b \leqslant c < n,\ 1 \leqslant d \leqslant n\}.$$

We have seen that r' solves the classical (modified) Yang–Baxter equation (59), and have identified it in terms of well known solutions of this equation. In particular, we have shown that in the hyperbolic and trigonometric cases the above r' with $\Omega = -\frac{1}{n}$ is equivalent to a multiple of the Cremmer–Gervais classical r-matrix under an automorphism of gl_n. We obtained these results by an explicit determination of the gauge transformations $g(q) \in GL_n$ for which the Poisson brackets of $L'(q,p) = g(q)L(q,p)g^{-1}(q)$, where L is the standard Lax matrix (6), can be written in the form (2) with a constant r-matrix. The gauge transformation $g(q)$ for which the Poisson brackets of L' are encoded by r' in (94) was found as the product

$$g(q) = \exp\left(-Xn\Omega\sum_{i=1}^{n} q_i\right)\varphi(q)\chi(q) \tag{95}$$

where the matrices $\varphi(q)$ and $\chi(q)$ are defined by (49) and (50), with the notation fixed by equations (7), (8) and (13) in section 2.

The outcome of our direct analysis of the degenerate Calogero–Moser models is consistent with the results obtained in [27, 28, 30] by different means. We hope to present a more detailed comparison with the elliptic case as well as an analogous study for other Lie algebras elsewhere.

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Appendix A. Proof of theorem 1

The proof given below relies on the general analysis of the momentum-independent Calogero–Moser r-matrices presented by Braden and Suzuki in [12]. We first specialize the relevant results of [12] to our case and then further elaborate them to obtain the statement of theorem 1.

Consider the Lax matrix in (6) with a function w in (7). Our task is to find the most general momentum-independent r-matrix, $r(q)$, which satisfies equation (2), i.e.

$$\{L_1, L_2\}(q,p) = [r_{12}(q), L_1(q,p)] - [r_{21}(q), L_2(q,p)]. \tag{A1}$$

Obviously, $r(q) = r_{12}(q) \in gl_n \otimes gl_n$ can be expanded in the form

$$r(q) = \sum_{i,j=1}^{n} r^{i,j}(q) H_i \otimes H_j + \sum_{\alpha\in\Phi}\sum_{i=1}^{n}\left(r^{i,\alpha}(q) H_i \otimes E_\alpha + r^{\alpha,i}(q) E_\alpha \otimes H_i\right)$$
$$+ \sum_{\alpha,\beta\in\Phi} r^{\alpha,\beta}(q) E_\alpha \otimes E_\beta. \tag{A2}$$

Since the functions w in (7) are odd (and thus $w_{-\alpha}(q) = -w_\alpha(q)$), we can use the results of the third and fourth chapters of [12], where it has been shown that under our conditions the following equations hold:

$$r^{\alpha,i}(q) = 0 \qquad (\forall i \in \{1,\dots,n\},\ \forall\alpha\in\Phi) \tag{A3}$$

$$r^{\alpha,\beta}(q) = \frac{w'_\alpha(q)}{w_\alpha(q)}\delta_{\alpha,-\beta} = -F_\alpha(q)\delta_{\alpha,-\beta} \qquad (\forall\alpha,\beta\in\Phi). \tag{A4}$$

Moreover, according to [12], the remaining requirements on $r(q)$ reduce to the equations

$$\sum_{i=1}^{n}\alpha_i r^{i,j}(q) = 0 \qquad (\forall\alpha\in\Phi,\ \forall j\in\{1,\dots,n\}) \tag{A5}$$

and

$$\sum_{i=1}^{n}(\alpha_i r^{i,\beta}w_\alpha - \beta_i r^{i,\alpha}w_\beta) = c^{\beta}_{-\alpha,\alpha+\beta}(r^{-\alpha,\alpha}w_{\alpha+\beta} + r^{-\beta,\beta}w_{\alpha+\beta}) \qquad (\forall\alpha,\beta\in\Phi). \tag{A6}$$

Here we use the basis of gl_n introduced in (5), $\alpha_i := \alpha(H_i)$, the structure constants $c^{\alpha+\beta}_{\alpha,\beta}$ satisfy $[E_\alpha, E_\beta] = c^{\alpha+\beta}_{\alpha,\beta}E_{\alpha+\beta}$ if α, β, $(\alpha+\beta)$ all belong to Φ, and $c^{\alpha+\beta}_{\alpha,\beta} := 0$ otherwise.

Now consider equation (A5) for $\alpha := (\lambda_k - \lambda_l) \in \Phi$. From this we see that $r^{k,j}(q) - r^{l,j}(q) = 0$ $(k \neq l, \forall j)$, which means that the general solution of (A5) is

$$r^{i,j}(q) = M^j(q) \qquad (\forall i,j\in\{1,\dots,n\}) \tag{A7}$$

where the M^j are arbitrary smooth functions of q. Let us next solve (A6) for $r^{i,\alpha}(q)$. By substituting (A4) into (A6) and using the identity (10) and the symmetry properties of the structure constants we obtain

$$\sum_{i=1}^{n}\left(\alpha_i\hat r^{i,\beta}(q) - \beta_i\hat r^{i,\alpha}(q)\right) = c^{\alpha+\beta}_{\alpha,\beta} \qquad (\forall\alpha,\beta\in\Phi) \tag{A8}$$

where we define $\hat r^{i,\gamma} := \frac{r^{i,\gamma}}{w_\gamma}$ for any $\gamma\in\Phi$. By introducing the notation

$$\hat r^\alpha_S := \sum_{i=1}^{n}(\hat r^{i,\alpha} + \hat r^{i,-\alpha})H_i \qquad \hat r^\alpha_A := \sum_{i=1}^{n}(\hat r^{i,\alpha} - \hat r^{i,-\alpha})H_i \tag{A9}$$

we have

$$\hat r^\alpha := \sum_{i=1}^{n}\hat r^{i,\alpha}H_i = \tfrac{1}{2}(\hat r^\alpha_S + \hat r^\alpha_A) \qquad \hat r^\alpha_S(q) = \hat r^{-\alpha}_S(q) \qquad \hat r^{-\alpha}_A(q) = -\hat r^\alpha_A(q). \tag{A10}$$

We now consider equation (A8) for the pairs of roots (α,β) and $(\alpha,-\beta)$. By adding these two equations we obtain

$$\alpha(\hat r^\beta_S(q)) = c^{\alpha+\beta}_{\alpha,\beta} + c^{\alpha-\beta}_{\alpha,-\beta} \qquad (\forall\alpha,\beta\in\Phi). \tag{A11}$$

It follows from the definition of K_α in (5) that $\alpha(K_\beta) = -(c^{\alpha+\beta}_{\alpha,\beta} + c^{\alpha-\beta}_{\alpha,-\beta})$ for any $\alpha,\beta\in\Phi$. Therefore, the general solution of (A11) is given by

$$\hat r^\alpha_S(q) = -K_\alpha + \tau^\alpha_S(q)\mathbf{1}_n \qquad (\forall\alpha\in\Phi) \tag{A12}$$

where $\tau^\alpha_S(q) = \tau^{-\alpha}_S(q)$ are arbitrary smooth functions. On the other hand, by substituting (A11) and the decomposition in (A10) into (A8) we obtain the relation

$$\alpha(\hat r^\beta_A(q)) = \beta(\hat r^\alpha_A(q)) \qquad (\forall\alpha,\beta\in\Phi). \tag{A13}$$

Obviously, there exists the decomposition

$$\hat{r}^\alpha_A(q) = C_\alpha(q) + \tau^\alpha_A(q)\mathbf{1}_n \tag{A14}$$

where $C_\alpha(q) \in \mathcal{H}_n \subset sl_n$ and $\tau^\alpha_A(q)$ are smooth functions. The antisymmetry of $\hat{r}^\alpha_A(q)$ in α and (A13) can be rewritten as

$$C_{-\alpha}(q) = -C_\alpha(q) \qquad \alpha(C_\beta(q)) = \beta(C_\alpha(q)) \qquad \tau^{-\alpha}_A(q) = -\tau^\alpha_A(q). \tag{A15}$$

By the above, we have parametrized the most general $r(q)$ in terms of the functions M^j, τ^α_A, τ^α_S and C_α. If we now introduce the notation

$$Q(q) := \sum_{i=1}^n M^i(q)H_i + \frac{1}{2}\sum_{\alpha\in\Phi}\left(\tau^\alpha_S(q) + \tau^\alpha_A(q)\right) w_\alpha(q)E_\alpha \tag{A16}$$

then $r(q)$ in (A2) takes precisely the form stated by theorem 1, which completes the proof.

Appendix B. Proof of proposition 2

In this appendix we prove proposition 2 by analysing equation (32),

$$\alpha_k w^2_\alpha \delta_{\beta,-\alpha} - c^{\alpha+\beta}_{\alpha,\beta}\frac{w_\alpha w_\beta}{w_{\alpha+\beta}} A^{\alpha+\beta}_k + (\alpha\cdot r^\beta)A^\alpha_k + (\beta\cdot A^\alpha)A^\beta_k = 0 \qquad (\forall k = 1,\dots,n) \tag{B1}$$

whereby we determine the constants b^α_k that appear in $A^\alpha_k = w_\alpha b^\alpha_k$ (33). Here we use the notation $\alpha\cdot r^\beta = \sum_{i=1}^n \alpha_i r^\beta_i$, $\beta\cdot A^\alpha = \sum_{i=1}^n \beta_i A^\alpha_i$ and similarly for all quantities with Cartan indices. For later reference, note from (25) that

$$\beta\cdot r^\alpha = \tfrac{1}{2} w_\alpha \beta\cdot(C_\alpha - K_\alpha) \qquad \forall\alpha,\beta\in\Phi \tag{B2}$$

where K_α is defined in (5) and $C_\alpha = \sum_{i=1}^n C^i_\alpha H_i$ enjoys the properties in (15).

If we fix $\alpha\in\Phi$, then (B1) for the pairs of roots (α,β) given by (α,α), $(-\alpha,-\alpha)$, $(\alpha,-\alpha)$ and $(-\alpha,\alpha)$ leads, respectively, to the following relations:

$$(\alpha\cdot r^\alpha + \alpha\cdot A^\alpha)A^\alpha_k = 0 \tag{B3}$$

$$(\alpha\cdot r^{-\alpha} + \alpha\cdot A^{-\alpha})A^{-\alpha}_k = 0 \tag{B4}$$

$$\alpha_k w^2_\alpha + (\alpha\cdot r^{-\alpha})A^\alpha_k - (\alpha\cdot A^\alpha)A^{-\alpha}_k = 0 \tag{B5}$$

$$\alpha_k w^2_\alpha + (\alpha\cdot r^\alpha)A^{-\alpha}_k - (\alpha\cdot A^{-\alpha})A^\alpha_k = 0. \tag{B6}$$

Since $\alpha\cdot r^\alpha = \alpha\cdot r^{-\alpha}$ by (B2), these relations imply that

$$\alpha\cdot A^\alpha = \alpha\cdot A^{-\alpha} = -\alpha\cdot r^\alpha. \tag{B7}$$

On account of (B7) and (B2), equation (B5) can be written as

$$\alpha_k w^2_\alpha = (\alpha\cdot A^\alpha)(A^\alpha_k + A^{-\alpha}_k). \tag{B8}$$

This expression shows that

$$b^\alpha_k - b^{-\alpha}_k = \varepsilon^\alpha \alpha_k \tag{B9}$$

with some constants ε^α. We then find from the above that

$$\alpha\cdot b^\alpha = \varepsilon^\alpha \tag{B10}$$

and the ε^α must satisfy

$$\varepsilon^\alpha = \varepsilon^{-\alpha} \qquad (\varepsilon^\alpha)^2 = 1. \tag{B11}$$

Now it is convenient to introduce $\Pi_k^\alpha := (b_k^\alpha + b_k^{-\alpha})$, which results in

$$b_k^\alpha = \tfrac{1}{2}\varepsilon^\alpha \alpha_k + \tfrac{1}{2}\Pi_k^\alpha \qquad \forall \alpha \in \Phi. \tag{B12}$$

Let us put $\Pi_k^{ij} := \Pi_k^{(\lambda_i - \lambda_j)}$. Then the relations $\Pi_k^\alpha = \Pi_k^{-\alpha}$ and $\alpha \cdot \Pi^\alpha = 0$ (by (B7)) give

$$\Pi_k^{ij} = \Pi_k^{ji} \qquad \Pi_i^{ij} = \Pi_j^{ij} \qquad \forall k,\ i \neq j. \tag{B13}$$

Consider now such roots $\alpha = (\lambda_i - \lambda_j)$ and $\beta = \pm(\lambda_l - \lambda_m) \in \Phi$ that $\{i, j\} \cap \{l, m\} = \emptyset$. In this case (B1) yields

$$(\alpha \cdot \hat{r}^\beta) b_k^\alpha + (\beta \cdot b^\alpha) b_k^\beta = 0 \tag{B14}$$

$$(\alpha \cdot \hat{r}^{-\beta}) b_k^\alpha - (\beta \cdot b^\alpha) b_k^{-\beta} = 0 \tag{B15}$$

where we use the notation $\hat{r}^\gamma := \frac{r^\gamma}{w_\gamma}$ for any $\gamma \in \Phi$. Adding these two equations, and using (B7) and (B12), we can easily find that now

$$\beta \cdot b^\alpha = 0 \qquad \beta \cdot \Pi^\alpha = 0. \tag{B16}$$

The general form of Π_k^{ij} which obeys (B13) and (B16) is, in fact, the following:

$$\Pi_k^{ij} = \eta^\alpha (\delta_{ki} + \delta_{kj}) + 2\Omega^\alpha \tag{B17}$$

where η^α, Ω^α are constants. Note that for $\alpha = (\lambda_i - \lambda_j)$ that element $K_\alpha = \sum_{k=1}^n K_\alpha^k H_k$ defined in (5) has precisely the components $K_\alpha^k = \delta_{ki} + \delta_{kj}$.

Now, let $\alpha, \beta, \alpha + \beta \in \Phi$ be roots. In this case $\alpha - \beta = \alpha + (-\beta) \notin \Phi$. Hence (B1) for the (α, β) and the $(\alpha, -\beta)$ pairs reads as

$$c_{\alpha,\beta}^{\alpha+\beta} b_k^{\alpha+\beta} = (\alpha \cdot \hat{r}^\beta) b_k^\alpha + (\beta \cdot b^\alpha) b_k^\beta \tag{B18}$$

$$0 = (\alpha \cdot \hat{r}^{-\beta}) b_k^\alpha - (\beta \cdot b^\alpha) b_k^{-\beta}. \tag{B19}$$

By adding these two equations, making use of (B2) and (B9), we obtain

$$c_{\alpha,\beta}^{\alpha+\beta} b_k^{\alpha+\beta} = -(\alpha \cdot K_\beta) b_k^\alpha + \varepsilon^\beta (\beta \cdot b^\alpha) \beta_k. \tag{B20}$$

If $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_j - \lambda_l)$ are chosen, then $\alpha \cdot K_\beta = -1$ and $c_{\alpha,\beta}^{\alpha+\beta} = 1$. Let us then substitute (B12) with (B17) into (B20) and consider the resulting equation for $k \notin \{i, j, l\}$ and for $k \in \{i, j, l\}$. In this way we obtain the requirements†

$$\Omega^{\alpha+\beta} = \Omega^\alpha \tag{B21}$$

$$\varepsilon^{\alpha+\beta} + \eta^{\alpha+\beta} = \varepsilon^\alpha + \eta^\alpha \tag{B22}$$

$$\varepsilon^\alpha - \eta^\alpha = 2\varepsilon^\beta (\beta \cdot b^\alpha) \tag{B23}$$

$$\eta^{\alpha+\beta} - \varepsilon^{\alpha+\beta} = -2\varepsilon^\beta (\beta \cdot b^\alpha). \tag{B24}$$

These tell us that

$$\Omega^{\alpha+\beta} = \Omega^\alpha \qquad \varepsilon^{\alpha+\beta} = \varepsilon^\alpha \qquad \eta^{\alpha+\beta} = \eta^\alpha. \tag{B25}$$

† Here we implicitly assume that $n \geqslant 4$, but the final solution is valid for any $n \geqslant 2$.

In conclusion, there exist some constants ε, η, Ω that

$$\varepsilon^\alpha = \varepsilon \qquad \eta^\alpha = \eta \qquad \Omega^\alpha = \Omega \qquad \forall\alpha \in \Phi. \tag{B26}$$

In addition, we can compute from (B12) that in the above case $2\beta \cdot b^\alpha = (\eta - \varepsilon)$, and by substituting this back into (B23) we obtain

$$(\varepsilon + 1)(\eta - \varepsilon) = 0. \tag{B27}$$

At the same time we know from (B11) that ε must be equal to 1 or -1.

The first solution of (B27) is $\varepsilon = 1 = \eta$. In this case we can determine b_k^α from (B12) in terms of the arbitrary constant Ω as

$$b_k^{\lambda_i - \lambda_j} = \delta_{ki} + \Omega. \tag{B28}$$

We can then also calculate $\beta \cdot r^\alpha$ from the above equations, and thereby find from (B2) that $C_\alpha = -H_\alpha$ must hold. This is precisely the result stated in case I of proposition 2. We have obtained it as a consequence of considering a subset of all cases of (B1), but it can checked to satisfy this equation in all remaining cases (for $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_l - \lambda_i)$, etc) as well.

The other solution of (B27) is $\varepsilon = -1$, but then we still have to determine η. For this we consider $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_j - \lambda_l)$ and calculate that

$$b_k^\alpha = \tfrac{1}{2}\left((\eta - 1)\,\delta_{ki} + (\eta + 1)\,\delta_{kj}\right) + \Omega \tag{B29}$$

$$b_k^\beta = \tfrac{1}{2}\left((\eta - 1)\,\delta_{kj} + (\eta + 1)\,\delta_{kl}\right) + \Omega. \tag{B30}$$

We then look at (B1) for the (α, β) and (β, α) pairs of roots and add these two equations, which gives

$$0 = (\alpha \cdot \hat{r}^\beta + \alpha \cdot b^\beta) b_k^\alpha + (\beta \cdot \hat{r}^\alpha + \beta \cdot b^\alpha) b_k^\beta. \tag{B31}$$

Since b_k^α and b_k^β are linearly independent n-component vectors for any η, we obtain

$$\alpha \cdot \hat{r}^\beta + \alpha \cdot b^\beta = 0 \qquad \beta \cdot \hat{r}^\alpha + \beta \cdot b^\alpha = 0. \tag{B32}$$

By subtracting these equations and taking into account that by (B2) now

$$\alpha \cdot \hat{r}^\beta - \beta \cdot \hat{r}^\alpha = \tfrac{1}{2}\left(\beta \cdot K_\alpha - \alpha \cdot K_\beta\right) = 1 \tag{B33}$$

we find that $\eta = 1$. So we have completely determined b_k^α again, and it is easy to confirm that the final formula agrees with case II of proposition 2. Thus the proof is complete.

Appendix C. Proof of proposition 5

In this appendix we verify the statement of proposition 5.

By combining equation (18) and proposition 4, the constant r-matrix that we wish to calculate can be written in the form

$$\tilde{r}' = (\varphi(q) \otimes \varphi(q))\, \rho(q)\, (\varphi(q) \otimes \varphi(q))^{-1} \tag{C1}$$

with

$$\rho(q) = (\chi(q) \otimes \chi(q)) \left(\tilde{r}(q) + \sum_k A_k(q) \otimes H_k\right) (\chi(q) \otimes \chi(q))^{-1}. \tag{C2}$$

The formulae in (48) and (50) together with (10) and (11) result in

$$\rho = -\mathcal{B}\sum_{k\neq l}\frac{1}{F_k - F_l}(e_{kl} - e_{ll}) \otimes (e_{lk} - e_{kk}) + \sum_{k\neq l}\frac{F_k F_l}{F_k - F_l}(e_{kl} - e_{ll}) \otimes (e_{lk} - e_{kk})$$
$$+ \sum_{k\neq l} F_k e_{kk} \otimes e_{kl} - \sum_{k\neq l} F_l e_{lk} \otimes e_{ll}. \tag{C3}$$

Therefore, to prove proposition 5 it is enough to verify that

$$(\varphi(q) \otimes \varphi(q))\,\rho(q) = \tilde{r}'\,(\varphi(q) \otimes \varphi(q)) \tag{C4}$$

holds for ρ in (C3) and $\tilde{r}'$ in (61). We obtain in a straightforward manner that

$$(\varphi(q) \otimes \varphi(q))\,\rho(q) = \sum_{a,b,c,d=1}^{n} \big(\mathcal{B}B_{abcd} + \tilde{B}_{abcd}\big) e_{ab} \otimes e_{cd} \tag{C5}$$

where

$$B_{abcd} = \frac{(\varphi_{ad} - \varphi_{ab})(\varphi_{cd} - \varphi_{cb})}{F_d - F_b} \qquad \text{if} \quad b \neq d \tag{C6}$$

$$\tilde{B}_{abcd} = \frac{F_d F_b}{F_d - F_b}(\varphi_{ad} - \varphi_{ab})(\varphi_{cd} - \varphi_{cb}) + F_d\varphi_{ad}\varphi_{cd} - F_b\varphi_{ab}\varphi_{cb} \qquad \text{if} \quad b \neq d \tag{C7}$$

and $B_{abcd} = \tilde{B}_{abcd} = 0$ if $b = d$. From (49) and (61), the right-hand side of (C4) is found to be

$$\tilde{r}'\,(\varphi(q) \otimes \varphi(q)) = \sum_{a,b,c,d=1}^{n} \big(\mathcal{B}D_{abcd} + \tilde{D}_{abcd}\big) e_{ab} \otimes e_{cd} \tag{C8}$$

with

$$D_{abcd} = \sum_{(a,x,c,y)\in S} \varphi_{xb}\varphi_{yd} - \sum_{(c,y,a,x)\in S} \varphi_{xb}\varphi_{yd} \tag{C9}$$

$$\tilde{D}_{abcd} = \sum_{(a-1,x,c-1,y)\in S} \varphi_{xb}\varphi_{yd} - \sum_{(c-1,y,a-1,x)\in S} \varphi_{xb}\varphi_{yd} \tag{C10}$$

where the set S is defined in proposition 5 and by an empty sum we mean zero.

We now observe that $\tilde{D}_{abcd} = 0 = \tilde{B}_{abcd}$ if $a = 1$ or $c = 1$, and

$$\tilde{D}_{a,b,c,d} = D_{a-1,b,c-1,d} \qquad \tilde{B}_{a,b,c,d} = B_{a-1,b,c-1,d} \qquad \text{if} \quad 2 \leqslant a, c \leqslant n. \tag{C11}$$

These properties are obvious for $\tilde{D}$, while for $\tilde{B}$ they follow from the formula (49). In particular, the second equality in (C11) is checked by inserting into (C6) the identity

$$\varphi_{a-1,d} - \varphi_{a-1,b} = F_b\varphi_{ab} - F_d\varphi_{ad} \qquad 2 \leqslant a \leqslant n \tag{C12}$$

which is a consequence of (49). We conclude that it is sufficient to show that $B_{abcd} = D_{abcd}$.

Let us examine the expressions for B_{abcd} and D_{abcd}. First, we note that for all indices

$$B_{abcd} = B_{cbad} \qquad D_{abcd} = D_{cbad} \tag{C13}$$

and

$$B_{abcd} = 0 = D_{abcd} \qquad \text{if} \quad a = n \qquad \text{or} \quad c = n \qquad \text{or} \quad b = d. \tag{C14}$$

Hence it is enough to show that $B_{abcd} = D_{abcd}$ for such indices that $a \leqslant c < n$ and $b \neq d$. We now introduce the notation

$$F_P := \prod_{t\in P} F_t \qquad \forall P \subset \{1, \ldots, n\} \tag{C15}$$

and also put $F_P := 1$ if $P = \emptyset$, for which $|P| = 0$. We then rewrite B_{abcd} as

$$B_{abcd} = (F_d - F_b)\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-1-a}} F_P\Bigg)\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-1-c}} F_P\Bigg) \tag{C16}$$

where I_k^n is defined in (47). This is derived from (C6) by using that as a result of (49)

$$\varphi_{al} - \varphi_{ak} = (F_k - F_l) \sum_{\substack{P\subset I_k^n\cap I_l^n\\ |P|=n-1-a}} F_P. \tag{C17}$$

Next, by inserting (49) into (C9) and using that $a \leqslant c$, we obtain the expression

$$\begin{aligned} D_{abcd} = (F_d - F_b) \sum_{\substack{x+y=a+c+1\\ 1\leqslant x\leqslant a<y\leqslant n}} &\Bigg(\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-1-x}} F_P\Bigg)\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-y}} F_P\Bigg) \\ &-\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-1-y}} F_P\Bigg)\Bigg(\sum_{\substack{P\subset I_b^n\cap I_d^n\\ |P|=n-x}} F_P\Bigg)\Bigg). \end{aligned} \tag{C18}$$

The $x = a$, $y = (c + 1)$ term in the first line of the right-hand side of (C18) clearly equals the right-hand side of (C16). The proof is completed by a close inspection of the ranges of the summation indices, which shows that all the remaining terms cancel pairwise between the two lines of (C18) for any $a \leqslant c \leqslant (n - 1)$.

References

[1] Calogero F 1975 *Nuovo Cimento Lett.* **13** 411
[2] Moser J 1975 *Adv. Math.* **16** 197
[3] Fehér L and Pusztai B G 2000 *Czech. J. Phys.* **50** 59
[4] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* vol 1 (Boston, MA: Birkhäuser)
[5] Olshanetsky M A and Perelomov A M 1976 *Invent. Math.* **37** 93
[6] d'Hoker E and Phong D H 1998 *Nucl. Phys.* B **530** 537
[7] Bordner A J, Corrigan E and Sasaki R 1998 *Prog. Theor. Phys.* **100** 1107
[8] Babelon O and Viallet C M 1990 *Phys. Lett.* B **237** 411
[9] Avan J and Talon M 1993 *Phys. Lett.* B **303** 33
[10] Krichever I 1980 *Funct. Anal. Appl.* **14** 282
[11] Sklyanin E K 1994 *St Petersburg Math. J.* **6** 227
[12] Braden H W and Suzuki T 1994 *Lett. Math. Phys.* **30** 147
[13] Avan J, Babelon O and Talon M 1995 *St Petersburg Math. J.* **6** 255
[14] Arutyunov G E and Medvedev P B 1996 *Phys. Lett.* A **223** 66
[15] Forger M and Winterhalder A 1999 New approach to the integrability of the Calogero models *Preprint* hep-th/9912109
[16] Avan J and Rollet G 1996 *Phys. Lett.* A **212** 50
[17] Nijhoff F W, Kuznetsov V B, Sklyanin E K and Ragnisco O 1996 *J. Phys. A: Math. Gen.* **29** L333
[18] Suris Y B 1997 *Phys. Lett.* A **225** 253
[19] Ruijsenaars S N M and Schneider H 1986 *Ann. Phys., NY* **170** 370
[20] Avan J, Babelon O and Billey E 1996 *Commun. Math. Phys.* **178** 281

[21] Felder G and Varchenko A 1997 *J. Stat. Phys.* **89** 963
[22] Arutyunov G E and Frolov S A 1998 *Commun. Math. Phys.* **191** 15
[23] Arutyunov G E, Chekhov L O and Frolov S A 1998 *Commun. Math. Phys.* **192** 405
[24] Etingof P and Schiffmann O 1999 Lectures on the dynamical Yang–Baxter equations *Preprint* math.QA/9908064
[25] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[26] Belavin A A 1981 *Nucl. Phys.* B **180** 189
[27] Hou B Y and Yang W L 1999 *J. Phys. A: Math. Gen.* **32** 1475
Hou B Y and Yang W L 2000 *J. Math. Phys.* **41** 357
[28] Hasegawa K 1997 *Commun. Math. Phys.* **187** 289
[29] Ruijsenaars S N M 1987 *Commun. Math. Phys.* **110** 191
[30] Antonov A, Hasegawa K and Zabrodin A 1997 *Nucl. Phys.* B **503** 747
[31] Cremmer E and Gervais J L 1990 *Commun. Math. Phys.* **134** 619
[32] Belavin A A and Drinfeld V G 1982 *Funct. Anal. Appl.* **16** 159
[33] Gerstenhaber M and Giaquinto A 1997 *Lett. Math. Phys.* **40** 337
[34] Endelman R and Hodges T J Quantization of certain skew-symmetric solutions of the classical Yang–Baxter equation *Preprint* math.QA/0003066